STABILITY IMPLICATIONS OF DELAY DISTRIBUTION FOR FIRST-ORDER AND SECOND-ORDER SYSTEMS

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Abstract. In application areas, such as biology, physics and engineering, delays arise naturally because of the time it takes for the system to react to internal or external events. Often a delay is not fixed but varies according to some distribution function. This paper considers the effect of delay distribution on the asymptotic stability of the zero solution of functional differential equations — the corresponding mathematical models. We first show that the asymptotic stability of the zero solution of a first-order scalar equation with symmetrically distributed delay follows from the stability of the corresponding equation where the delay is fixed and given by the mean of the distribution. This result completes a proof of a stability condition in [Bernard, S., Bélair, J. and Mackey, M. C. Sufficient conditions for stability of linear differential equations with distributed delay. *Discrete Contin. Dyn. Syst. Ser. B*, 1(2):233–256, 2001], which was motivated in turn by an application from biology. We also discuss the corresponding case of second-order scalar delay differential equations, because they arise in physical systems that involve oscillating components. An example shows that it is not possible to give a general result for the second-order case. Namely, the boundaries of the stability regions of the distributed-delay equation and of the mean-delay equation may intersect, even if the distribution is symmetric.

1. Introduction

Ordinary differential equations are the mathematical models of deterministic processes in which the rate of change of the state variables, at any given time, depends on the state of the variables at that specific time. However, in many systems the relationship between a state variable and its rate of change is not so straight forward, because the response of the system depends not only on the present state of the system. A formal mathematical model of such a systems is called a *functional differential equation* (FDE). Furthermore, an FDE is called a *retarded*

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functional differential equation (RFDE) or *delayed differential equation* (DDE) if the response is a function of the past values of one or more state variables. The specific dependence on the states in the past may be quite complicated. Mathematical models in the form of RFDEs can be found in many different fields of science, including biology, physics and engineering; see, for example, the text books [23] and [18] as entry points to the literature.

In the simplest type of RFDE the rate of change of the state variables, at any given time $t \in \mathbb{R}^+$, depends on the state at a prior time. That is, the equation is given in the form of a DDE with a single fixed delay. Laser systems with optical feedback from external components are examples of systems that can be modelled in this way, where the delay time arises as the travel time of light between laser and component [15, 16]. Another class of examples are models for machine tool vibration, where the fixed delay arises from the fixed rotation speed of the tool or the work piece [7, 24, 25]. However, in many situations the process under investigation requires a mathematical description with more than one delay. While the linear stability analysis of equations with two delays is still feasible [11], the qualitative analysis of the equation in question is getting harder with a growing number of delays. Already the study of DDEs with three delays is quite challenging.

One way of dealing with the problem of many delays is to consider the case that the delays are distributed as given by a density function; or more generaly, by a distribution function. In fact, in many applications, for example, in mathematical biology, this is a natural modelling assumption. Specifically, models with distrubuted delays have been considered in population dynamics [8] and in neural modelling [5]; see also [6] and further references therein. An important question is the stability of a given steady state solutions. According to well-established theory of functional differential equations [13], information about the local dynamics of a nonlinear RFDE can be obtained from the linearization about the steady state. Hence, one needs to consider the stability properties of the linearized system.

An autonomous linear scalar RFDE takes the general form

$$\dot{x}(t) = \int_0^r x(t-\theta)d\eta(\theta) \tag{1}$$

where $\eta(\theta),\ 0 \le \theta \le r$, is of bounded variation so that η is continuous from the right on $(0, r)$. Here the function η can be considered as a distribution function, possibly after a normalization. In this formulation one can consider not only absolutely continuous but also discrete distribution functions. Therefore, in an RFDE with at most countably many delays, the associated delay distribution function is given as a step function, so that there is no need to model discrete delays by Dirac delta functions.

In this paper we consider what can be said about the relationship between the qualitative properties of the distributed-delay equations and the single fixed delay DDE where the delay is given by the *expectation value* (or mean) of the delay distribution function η. More specifically, we investigate for first-order and second-order scalar equations what can be learnt about the linear stability of the distributed-delay equation from the linear stability of the mean-delay equation, where the delay is the mean of the distribution. The emphasis here is on stability results that hold for any delay distribution with a given mean, without assuming any other properties (apart from symmetry of the distribution around its mean). This approach is not only quite natural mathematically, but also of interest from the applications point of view. Namely, in many concrete systems under investigation the delay distribution is actually unknown, but measuring its mean (or even some higher order moments) might be possible; see also [6] for delay distribution independent stability results in the same spirit.

We first investigate the following first-order equation

$$\dot{x}(t) = -ax(t) - b\int_0^h x(t-\tau)d\mu(\tau), \tag{2}$$

where $a, b \in \mathbb{R}$, the integral is of Stieltjes-type, and $\mu : \mathbb{R} \to \mathbb{R}$ is a non-decreasing and right-continuous function satisfying

(A1) $\mu(\tau) = 1,\ if\ \tau \geq h$ and

(A2) $\mu(\tau) = 0,\ if\ \tau < 0$,

for $h \geq 0$. Conditions (A1) and (A2), together with the monotonicity of μ, imply that

$$\int_0^h d\mu(\tau) = 1.$$

The associated first-order mean-delay DDE is given by

$$\dot{x}(t) = -ax(t) - bx(t-E), \tag{3}$$

where $E > 0$ is the expectation value of the density function μ. Note that (3) is a special case of (2) for the special choice of distribution function given by

$$\mu_s(x) = \begin{cases} 1, & if\ x \geq E; \\ 0, & if\ x < E. \end{cases} \tag{4}$$

The goal is now to compare the stability regions in the (a, b)-plane of (3) and (2), where the zero solution $x \equiv 0$ is asymptotically stable. Specifically, the idea is to determine the stability of the distributed-delay equation from that of the mean-delay equation. Our motivation and starting point are the stability conditions for the first-order case stated in [2], where a model with distributed delays proposed in [14] to describe neutrophil dynamics served as a particular example. Our main result on first-order equations, formulated as Theorem 3.8 below, states that the trivial (or zero) solution of the distributed-delay equation (2) is asymptotically stable in the stability region of the mean-delay equation (3) if μ is symmetric, that is, it satisfies $\mu(E-x) = 1 - \mu(E+x-0)$. A statement such as this is very useful in applications, because it is always easier to check properties of equations with a single delay (in this case given by the mean E of μ). The statement of our theorem was used, but not proved, in the proof of Statement 2 of Theorem 4.0.5 in [2], where a stability condition is given. To keep this paper self-contained, we formulate this result as Theorem 3.9 below and show how the stability condition arises.

In several areas of application one encounters second-order systems in the form of mass-spring-damper oscillators subject to delays. Examples are models of machine tool vibrations, where the tool may oscillate relative to the work piece [7, 24, 25]. The main motivating example for us is the relatively new field of *substructuring* or *hybrid testing* of engineering structures, where an overall system is split into a critical part that is tested in the laboratory while the remainder of the system is run as a mathematical model on a computer. The tested part provides measured input into the computer model, which in turn generates input into the actual laboratory test via a transfer system that is usually implemented by hydraulic or electric actuators. Both the computer model and the transfer system result in an inevitable time delay when the whole loop of laboratory testing and computation is closed. Delay effects in hybrid testing raise interesting mathematical questions, and the systematic application of the theory of FDEs and RFDEs is an ongoing area of research in this field; see, for example, [1, 4, 10, 20, 26]. In many situations it would be quite natural to model the delay in a hybrid test as given by a distribution function.

This is why we consider the second-order case given by

$$\ddot{x}(t) = -\dot{x}(t) - ax(t) - b\int_0^h x(t-\tau)d\mu(\tau), \tag{5}$$

where the function μ in (5) satisfies the same conditions as before. The question is again how the stability region of the zero solution of (5) is related to that of the mean-delay equation

$$\ddot{x}(t) = -\dot{x}(t) - ax(t) - bx(t-E), \tag{6}$$

where E is the expectation value of μ. (Again, (6) is the special case of (5) for the distribution function μ_s from (4).) As it turns out, there is no equivalent statement as for the first-order. Namely, as we demonstrate with an example, in general, the stability region of the mean-delay equation (6) is not contained in that of the distributed-delay equation (5). On the other hand, experience suggests the following conjecture: when $E > 0$ is fixed then there is a first intersection for $a = a^*$ of the two stability regions. As a consequence, for a from the interval $[-1/E, a^*]$ and for any b stability of the zero solution of (6) would imply stability of the zero solution of equation (5) for any symmetric distribution function μ with expectation value E. This statement would constitute a step towards a general description of the stability of this type of system. A difficulty from the practical point of view is that the bound a^* may be small and hard to determine.

The outline of the paper is as follows. In Section 2 we introduce notation and recall some facts on RFDEs. In Section 3 we present the stability criterion for the first-order case, where we concentrate on the proof of Theorem 3.8 but also state the overall result, Theorem 3.9. The case of second-order equations is treated in Section 4. Finally, Section 5 summarizes and outlines future work.

2. Background and notation

In this paper we are dealing with linear autonomous equations, which take the general form

$$\dot{x}(t) = \int_0^r d\eta(\theta)x(t-\theta) \tag{7}$$

where $\eta(\theta)$, $0 \le \theta \le r$, is an $n \times n$ matrix of normalized functions of bounded variation so that η is continuous from the right on $(0, r)$ and $\eta(r) = 1$.

We now recall some facts of the general theory of RFDEs; see[9, 13]. A solution $x : \mathbb{R}^n \to \mathbb{R}^n$ of (7), for a given η and $h > 0$, is a differentiable function satisfying (7). Let $C = C([-h, 0], \mathbb{R}^n)$ denote the Banach space of continuous functions mapping the interval $[-h, 0]$ into $\mathbb{R}^n$, with the supremum norm, and define $x_t \in C$ as $x_t(\theta) = x(t+\theta)$, $\theta \in [-h, 0]$. A solution $x^\phi : [0, \infty) \to \mathbb{R}$ is uniquely determined with $x_0^\phi = \phi \in C$. The unique solution with initial function $\phi \in C$ determines a map

$$F(t, \phi) : \mathbb{R}^+ \times C \ni (t, \phi) \mapsto x_t^\phi \in C$$

and the *solution operator* is

$$T(t)\phi : C \ni \phi \mapsto F(t, \phi) \in C, \ t \ge 0.$$

The solution operator is a strongly continuous semigroup with an infinitesimal generator A, the spectrum $\sigma(A) \subset \mathbb{C}$ of which is formed by its point spectrum. Furthermore, for a $\lambda \in \sigma(A)$ if and only if λ satisfies the *characteristic equation*

$$\det\left(\lambda I - \int_0^h e^{-\lambda\theta} d\eta(\theta)\right) = 0. \tag{8}$$

The roots of (8) are called *characteristic roots*. The trivial solution of equation (7) is asymptotically stable if and only if the real part of all characteristic root of (8) is negative; see [13]. Thus local stability investigations can be carried out by locating the zeros of the *characteristic function*

$$\Delta(\lambda) : \mathbb{C} \ni \lambda \mapsto \det\left(\lambda I - \int_0^h e^{-\lambda\theta} d\eta(\theta)\right) \in \mathbb{C}.$$

This is a far from simple task since the characteristic function is an analytic function possessing countably infinitely many roots. A lot of examples of use and applications of stability analysis based on characteristic roots can be found in [23]. The corresponding function and equation related to (2) are

$$h(s) :\ \mathbb{C} \to \mathbb{C},\ s \mapsto s + a + b\int_0^h e^{-s\tau} d\mu(\tau) \tag{9}$$

and

$$s + a + b\int_0^h e^{-s\tau} d\mu(\tau) = 0,\ s \in \mathbb{C}. \tag{10}$$

Another method to examine the stability properties of functional differential equations is the use of Lyapunov functionals; related theorems and examples can be found in [13]. So called Razumikhin-type theorems give the opportunity to use Lyapunov functions instead of Lyapunov functionals in the stability analysis of functional differential equations, and they can be found in [13] as well. The following important statement was proven by this method.

Theorem 2.1. [17], *The trivial solution $x \equiv 0$ of the equation*

$$\dot{x}(t) = -\int_0^h x(t-\tau) d\mu(\tau),\ \ (h \in [0,\infty))$$

is asymptotically stable if

$$\int_0^h \tau d\mu(\tau) < \frac{\pi}{2}.$$

This result is useful for comparing different stability results formulated in terms of the expectation value E of the distribution function μ, when $a = 0$. Furthermore, this result is a generalization of the fact that the trivial solution of the equation

$$\dot{x}(t) = -bx(t-E) \tag{11}$$

is asymptotically stable if $0 < bE < \frac{\pi}{2}$. Here and throughout, for a function $\mu : \mathbb{R} \to \mathbb{R}$ satisfying (A1) and (A2) the *expectation value* E of μ is defined as usual as

$$E = \int_0^h \tau d\mu(\tau), \tag{12}$$

where we assume that the integral is finite. Notice that

$$0 \leq \int_0^h \tau d\mu(\tau) = E \leq h \int_0^h d\mu(\tau) = h.$$

To formulate our statements we need some definitions.

Definition 2.2. Let $\mu : \mathbb{R} \to \mathbb{R}$ be a monotonically nondecreasing function with expection value E. We say that μ is *symmetric about its mean* E if

$$\mu(E - x) = 1 - \mu(E + x - 0) \tag{13}$$

We remark that if $y = E - x,\ y \in [0, 2E]$ then

$$\mu(y) = 1 - \mu(2E - y - 0),\ y \in [0, 2E].$$

Considering $2E = h$, we get $\mu(y) = 1 - \mu(h - y - 0),\ y \in [0, h]$. Furthermore, $\mu(\tau) = 0$, if $\tau < 0$ and $\mu(\tau) = 1$, if $\tau \geq h$.

The following lemma is a useful tool in the stability analysis of parameter-dependent systems; throughout $\mathrm{Re}(\lambda)$ and $\mathrm{Im}(\lambda)$ denote the real and imaginary parts of a $\lambda \in \mathbb{C}$, respectively.

Lemma 2.3. [19] *Let $f(\lambda, \alpha) = \lambda^n + g(\lambda, \alpha)$ be an analytic function with respect to λ and α, where $\alpha \in \mathbb{R}^m$ and $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > -\beta$ for a positive constant β. Assume that*

$$\limsup\{|\lambda^{-n} g(\lambda, \alpha)| : \mathrm{Re}(\lambda) \geq 0,\ |\lambda| \to \infty\} < 1.$$

Then, as α varies, the sum of the roots of $f(\lambda, \alpha) = 0$ in the open right half-plane can change only if a root appears on or crosses the imaginary axis.

We now summarize a method that can be found in [9, Chapter 11], because we will apply it to (2) and (5). For notational convenience, we introduce as in [2]

$$C(\omega) = \int_0^h \cos(\omega\tau) d\mu(\tau)$$

and

$$S(\omega) = \int_0^h \sin(\omega\tau) d\mu(\tau).$$

Let $\Omega = \{\omega : S(\omega) = 0\}$ be the *zero-set* of $S(\omega)$. It is easy to see that $\Omega \neq \emptyset$. If we consider $S(\omega)$ and $C(\omega)$ as functions of $\omega \in \mathbb{R}$, both of them are of period $p > 0$. Thus, if $\omega \in \Omega$ then $kp\omega \in \Omega,\ k \in \mathbb{N}$. Notice that, considering an interval $I = (\omega_k, \omega_{k+1}),\ \omega_k, \omega_{k+1} \in \Omega$, such that $\omega_{k+1} = p + \omega_k$, there are two cases. Either there is no $\omega \in \Omega$ such that $\omega \in I$ or there is at least one $\omega \in \Omega$ such that $\omega \in I$. Let I_k be given by

$$I_k = \left((2k-1)\frac{p}{2}, (2k+1)\frac{p}{2}\right).$$

Let $I^+_{k,l}$ and $I^-_{k,m},\ 1 \leq l \leq i,\ 1 \leq m \leq j$ denote the subintervals in I_k such that $S(\omega) > 0$ and $S(\omega) < 0$ on them, and i and j are the numbers of these subintervals, respectively. We remark that i and j both depend on k, but we do not explicitly indicate this dependence in what follows to avoid even more complicated notation.

Separating the real and imaginary part of (10) when it has a pair of roots in the form of $\pm i\omega,\ \omega > 0$, we get the equation

$$a + b \int_0^h \cos(\omega\tau) d\mu(\tau) = 0 \tag{14}$$

and

$$\omega - b\int_0^h \sin(\omega\tau)d\mu(\tau) = 0, \tag{15}$$

thus, we can define the functions

$$a(\omega) : I \to \mathbb{R},\ \omega \mapsto -\omega\frac{C(\omega)}{S(\omega)}, \tag{16}$$

$$b(\omega) : I \to \mathbb{R},\ \omega \mapsto \frac{\omega}{S(\omega)}, \tag{17}$$

where I is one of the intervals $I^+_{k,l}$ and $I^-_{k,m}$, $1 \le l \le i,\ 1 \le m \le j$. These functions are even, so our attention can be restricted to $\omega \ge 0$.

Finally, we define the curves

$$\Gamma^+_{k,l} = \left\{(a(\omega), b(\omega))\,\middle|\,\omega \in I^+_{k,l}\right\} \text{ and } \Gamma^-_{k,m} = \left\{(a(\omega), b(\omega))\,\middle|\,\omega \in I^-_{k.m}\right\}. \tag{18}$$

Because of their definition, at any point of these curves equation (10) has a pair of purely imaginary roots, which are called the *critical roots* in [9]. To describe the effect of the parameters on these roots we introduce functions

$$F(a,b;s) : \mathbb{R}^2 \times \mathbb{C} \ni (a,b;s) \mapsto \mathrm{Re}\left(s + a + b\int_0^h e^{-s\tau}d\mu(\tau)\right) \in \mathbb{R},$$

$$G(a,b;s) : \mathbb{R}^2 \times \mathbb{C} \ni (a,b;s) \mapsto \mathrm{Im}\left(s + a + b\int_0^h e^{-s\tau}d\mu(\tau)\right) \in \mathbb{R}$$

and the matrix

$$M = \begin{pmatrix} D_aF & D_bF \\ D_aG & D_bG \end{pmatrix}\Bigg|_{(a,b;s)=(a_0,b_0,i\omega_0)}, \tag{19}$$

where (a_0, b_0) is a point on one of the curves definied in (18) and ω_0 is the corresponding parameter value. The determinant of M determines the behaviour of the critical roots, depending on two parameters, in the complex plane. Namely we have the following.

Theorem 2.4 ([9, Chapter 11, Proposition 2.13])**.** *The critical roots are in the parameter region to the left of the curve $(a(\omega), b(\omega))$, when we follow this curve in the direction of increasing ω, whenever* $\det M < 0$ *and to the right when* $\det M > 0$.

We remark that for first-order equations (19) takes the form

$$M = \begin{pmatrix} 1 & C(\omega) \\ 0 & -S(\omega) \end{pmatrix}. \tag{20}$$

Thus, because of Theorem 2.4, the sign of $S(\omega)$ determines the behaviour of the critical roots. In particular, the sign of $\sin(\omega E)$ does so when the delay distribution is given by (4).

With the following notion, we define an order (denoted by the symbol $\prec$) on a collection of non-intersecting plane curves, where our interest is in curves $\Gamma^\pm$ as defined above. We consider the *graph*

$$\mathrm{Gr}(\Gamma) = \{(f(x), g(x)) : x \in I\}$$

of a curve

$$\Gamma(x) : I \to (f(x), g(x)) \in \mathbb{R}^2$$

defined on an interval $I \subset \mathbb{R}$. Consider now two curves $\Gamma_1 = \{(f_1(x), g_1(x)) : x \in I_1\}$ and $\Gamma_2 = \{(f_2(x), g_2(x) : x \in I_2\}$ on I_1 and I_2, respectively, and such that $\mathrm{Gr}(\Gamma_1) \cap \mathrm{Gr}(\Gamma_2) = \emptyset$. Then Γ_1 is said to be *below* Γ_2 — denoted $\Gamma_1 \prec \Gamma_2$ — if there are $x_1 \in I_1$ and $x_2 \in I_2$ such that $f_1(x_1) = f_2(x_2)$ and $g_1(x_1) < g_1(x_2)$. Alternatively, we say that Γ_2 is above Γ_1.

3. Stability properties of first-order equations

In this section we state and prove our main result on first-order equations. We start by considering the stability diagram for the fixed delay case as give by (3). We then consider the distributed-delay equation (2). We first present some delay distribution-independent results before considering symmetrically distributed delays.

3.1. **Stability of the fixed delay equation.** For equation (3) the two functions $C(\omega) = \cos(\omega E)$ and $S(\omega) = \sin(\omega E)$ are of period $2\pi/E$; and $\Omega = \{k\pi/E : k \in \mathbb{N}\}$. In addition, $i = j = 1$, so that we can simplify the notation. That is, the corresponding intervals are $I_k^- = ((2k-1)\pi/E, 2k\pi/E)$, $I_k^+ = (k2\pi/E, (2k+1)\pi/E)$, $\omega_k, \omega_{k+1} \in \Omega$. Thus

$$a(\omega) : I \to \mathbb{R},\ \omega \mapsto -\omega \frac{\cos(\omega E)}{\sin(\omega E)},$$
$$b(\omega) : I \to \mathbb{R},\ \omega \mapsto \frac{\omega}{\sin(\omega E)},$$

that is

$$\Gamma_k^\pm = \left\{ \left(-\omega \frac{\cos(\omega E)}{\sin(\omega E)}, \frac{\omega}{\sin(\omega E)} \right) \Big| \omega \in I_k^\pm \right\}.$$

Note that $\mathrm{Gr}(\Gamma_0^+) = \mathrm{Gr}(\Gamma_0^-)$. If $b = -a$, then $\lambda = 0$ is a characteristic root. From the construction of $\Gamma_k^\pm$ it is clear that, apart from the points of these curves, there are no parameter values (a, b) such that (10) has a pair of purely imaginary root. Thus, because of Lemma 2.3, the number of roots with positive real part can change by crossing these curves. If we set $a = 0$ in (2) then Theorem 2.1 for the special choice of $\mu = \mu_s$ implies that there is no root for parameter pairs below Γ_0 and above the line $b = -a$. Thus the curve $\Gamma_0^+ = \{(a(\omega), b(\omega)) | \omega \in I_0^+\}$, together with the line $b = -a$, forms the boundary of the stability region in the parameter (a, b)-plane, where $x \equiv 0$ of equation (3) is asymptotically stable. Figure 1 shows the curves $\Gamma_k^\pm$ for $k = 0, 1, 2$ and the line $b = -a$ in the (a, b)-plane for $E = 1$. The figure illustrates that, for $i < j$, one has $\Gamma_i^+ \prec \Gamma_j^+$ and $\Gamma_j^- \prec \Gamma_i^-$, which is a fact that is not hard to prove. The number of unstable characteristic roots is indicated for each region; the trivial solution is stable in the grey shaded region. This stability region of this first-order equation is well known; see, for example, [9, 12].

Via a simple rescaling of time (3) takes the form

$$\frac{1}{E}\dot{x}(t) = -ax(t) - bx(t-1). \tag{21}$$

Assuming a pair of purely imaginary roots $\pm i\omega$ with $\omega > 0$ in the corresponding characteristic equation, one obtains

$$aE = -bE\cos(\omega), \tag{22}$$
$$\omega = bE\sin(\omega). \tag{23}$$

Squaring and adding the last two results at $\omega = E\sqrt{b^2 - a^2}$, we obtain the following stability condition.

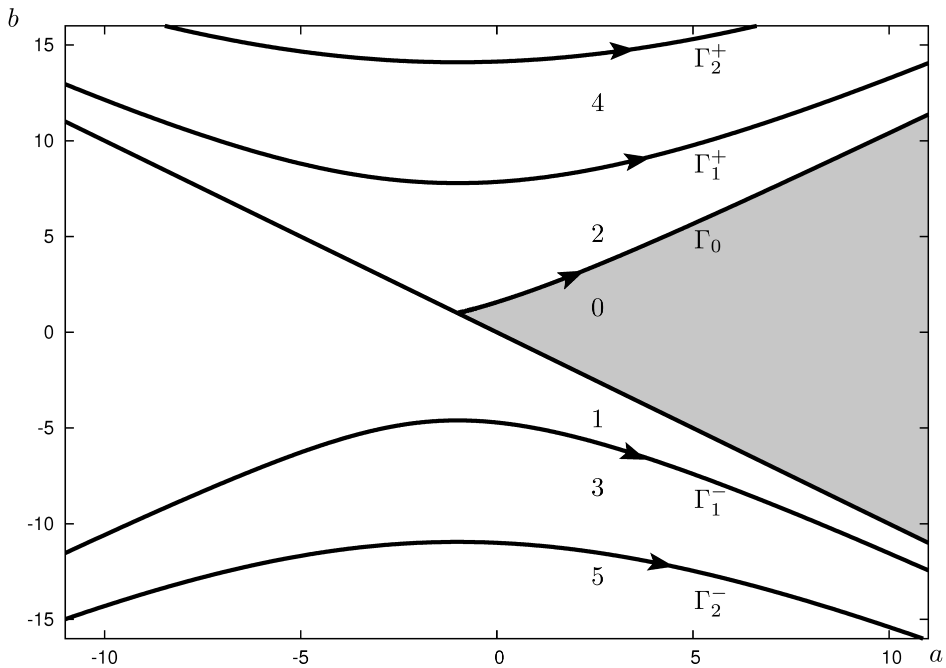

FIGURE 1. The curves $\Gamma_k^{\pm}$ for $k = 0, 1, 2$ and the line $b = -a$ of (3) for $E = 1$. The direction of increasing ω is indicated by the arrows and the number of characteristic roots with positive real parts are indicated in each region; the shaded part of the image is the stability region of the zero solution.

Proposition 3.1. *If $b > |a|$ in (3) then its zero solution is asymptotically stable if*

$$E < \frac{\arccos\left(-\frac{a}{b}\right)}{\sqrt{b^2 - a^2}}.$$

3.2. **Stability properties for a general delay distribution.**

Proposition 3.2. *If $b \leq -a$ in equation (2) then the trivial solution $x \equiv 0$ of (2) is not asymptotically stable.*

Proof. Let a be a fixed number. First, let us suppose that $b = -a$. Then $h(0) = 0$ because of (3).

Now, let us suppose that $b < -a$. We show that equation (10) has a root with positive real part. Let $s = \nu$ ($\nu \in \mathbb{R}$) and restrict (10) to the real line, that is, consider the function

$$\ell : \mathbb{R} \to \mathbb{R}, \ \nu \mapsto \nu + a + b \int_0^h e^{-\nu\tau} d\mu(\tau).$$

We have $\ell(0) = a + b < 0$. The continuity of $\ell(\nu)$ together with the fact that

$$\lim_{\nu \to \infty} \ell(\nu) = \infty$$

implies that there exists a $\nu^* > 0$ such that

$$\ell(\nu^*) = 0.$$

□

All possible values of $\omega \in \mathbb{R}$ for which $i\omega$ can be a root of (10) are described by the following.

Proposition 3.3. [2] *Let us suppose that $s = i\omega$ ($\omega \in \mathbb{R}$) in equation (10). Then $|b| \geq |a|$ and $|\omega| \leq \sqrt{b^2 - a^2}$.*

Proof. Taking real and imaginary parts, the characteristic equation can be written in the form

$$\mathrm{Re}(h(i\omega)) + i\mathrm{Im}(h(i\omega)) = 0.$$

Hence, a complex number $i\omega$ ($\omega \in \mathbb{R}$) can be a root of the characteristic equation if ω satisfies the equations

$$a + b\int_0^h \cos(\omega\tau)d\mu(\tau) = 0,$$

$$\omega - b\int_0^h \sin(\omega\tau)d\mu(\tau) = 0.$$

From these equations, we get the estimate

$$\begin{aligned}
\left(b\int_0^h \cos(\omega\tau)d\mu(\tau)\right)^2 + \left(b\int_0^h \sin(\omega\tau)d\mu(\tau)\right)^2 &\leq \\
b^2\int_0^h \cos^2(\omega\tau)d\mu(\tau) + b^2\int_0^h \sin^2(\omega\tau)d\mu(\tau) &= \\
b^2\int_0^h d\mu(\tau) &= b^2.
\end{aligned}$$

Thus we get $a^2 + \omega^2 \leq b^2$, that is $|b| \geq |a|$ and $|\omega| \leq \sqrt{b^2 - a^2}$. □

Proposition 3.4. *Let us suppose that $b = a > 0$ in equation (2). Then its zero-solution is asymptotically stable.*

Proof. We use the function

$$\nu_p : \mathbb{R} \to \mathbb{R},\ \tau \mapsto \begin{cases} \mu\left(\frac{\tau}{p}\right), & if\ 0 < p \leq 1, \\ \nu_0(\tau), & if\ p = 0, \end{cases} \tag{24}$$

where

$$\nu_0 : \mathbb{R} \to \mathbb{R},\ \tau \mapsto \begin{cases} 1, & if\ \tau \geq 0, \\ 0, & if\ \tau < 0, \end{cases}$$

and consider the parametrized equation

$$\dot{x}(t) = -ax(t) - b\int_0^{ph} x(t-\tau)d\nu_p(\tau)\ (a, b \in \mathbb{R}),\ p \in [0, 1]. \tag{25}$$

The function ν_p is monotonically increasing, right-continuous and satisfies assumptions (A1) and (A2). If $p = 0$ then, using equation (3), the characteristic equation of equation (25) is $s + a + b = 0$. Since, we assumed that $b = a > 0$, we get $s = -b - a < 0$. This proves our statement if $p = 0$.

Now, let $p \in (0, 1]$ and, to arrive at a contradiction, let us suppose that the trivial solution $x \equiv 0$ of equation (2) is not asymptotically stable. Then equation

(10) has a root $s \in \mathbb{C}$ such that $\mathrm{Re}(s) \geq 0$. Lemma 2.3 guarantees that there exists a parameter $p_0 \in [0,1]$ such that the characteristic equation of equation

$$\dot{x}(t) = -ax(t) - b\int_0^{p_0 h} x(t-\tau)d\nu_{p_0}(\tau)$$

has a root $i\omega$, with $\omega \in \mathbb{R}$. Proposition 3.3 implies that $\omega = 0$. But, if we put $s = 0$ into the characteristic function of the equation (25), then we get that

$$0 + a + b\int_0^{ph} d\nu_p(\tau) > 0$$

for all $p \in [0,1]$. □

From now on we assume that $b > |a|$, and remark that then $s = 0$ cannot be a characteristic root, because $a + b\int_0^h d\mu(\tau) = a + b \neq 0$.

3.3. **Stability for symmetrically distributed delays.** First we prove three lemmas that will be used in the proof of the main result.

Lemma 3.5. *Suppose that $\mu : \mathbb{R} \to \mathbb{R}$ is a right-continuous, monotonically nondecreasing function and that it is symmetric about its positive mean E. Then*

(1) for all $\omega \in \mathbb{R}$,

$$\left|\int_0^h \cos(\omega\tau)d\mu(\tau)\right| \leq |\cos(\omega E)|;$$

(2) if $0 \leq \omega < \frac{\pi}{2E}$, then

$$\int_0^h \cos(\omega\tau)d\mu(\tau) > 0.$$

Proof. We begin with statement 1 and first prove it for the case that μ is a step function. We introduce the set

$$S := \{s | \mu(E-s) - \mu(E-s-0) > 0,\ s \in (0,E]\}$$

and the function

$$\delta(x) : [0,E] \to \mathbb{R},\ x \mapsto \mu(E-x) - \mu(E-x-0).$$

Then

$$\begin{aligned}
\left|\int_0^h \cos(\omega\tau)d\mu(\tau)\right| &= \left|\delta(0)\cos(\omega E) + \sum_{s\in S}\delta(s)(\cos(\omega(E-s)) + \cos(\omega(E+s)))\right| \\
&= \left|\delta(0)\cos(\omega E) + \sum_{s\in S}2\delta(s)(\cos(\omega E)\cos(\omega s)\right| \\
&= \left|\delta(0)\cos(\omega E) + \cos(\omega E)\sum_{s\in S}2\delta(s)\cos(\omega s)\right| \\
&\leq \delta(0)\,|\cos(\omega E)| + |\cos(\omega E)|\sum_{s\in S}2\delta(s)|\cos(\omega s)| \\
&\leq |\cos(\omega E)|\left(\delta(0) + 2\sum_{s\in S}\delta(s)\right) \\
&= |\cos(\omega E)|.
\end{aligned}$$

Now let μ be an arbitrary function having the required properties, and let us take a sequence $\{\mu_n(x)\}_{n=1}^{\infty}$ of step functions. We assume that all of the μ_n satisfy the assumptions of the Lemma 3.5 and that

$$\lim_{n\to\infty} \mu_n(x) = \mu(x).$$

Furthermore, let

$$E_n = \int_0^h \tau d\mu_n(\tau).$$

Then

$$\lim_{n\to\infty} \left| \int_0^h \cos(\omega\tau) d\mu_n(\tau) \right| = \left| \int_0^h \cos(\omega\tau) d\mu(\tau) \right|, \tag{26}$$

$$\lim_{n\to\infty} \int_0^h \tau d\mu_n(\tau) = \int_0^h \tau d\mu(\tau),$$

namely

$$\lim_{n\to\infty} E_n = E.$$

Therefore,

$$\lim_{n\to\infty} |\cos(\omega E_n)| = |\cos(\omega E)|. \tag{27}$$

This shows that our statement is true if μ is a step function, so for all $n \in \mathbb{N}$

$$\left| \int_0^h \cos(\omega x) d\mu_n(x) \right| \leq |\cos(\omega E_n)|.$$

Equation (26) and (27) together with the fact that the limit is monotone implies that

$$\left| \int_0^h \cos(\omega x) d\mu(x) \right| \leq |\cos(\omega E)|.$$

This finishes the proof of the first statement.

Since μ is symmetric, it is obvious that $h = 2E$. Likewise, the symmetry implies that

$$\begin{aligned} \int_0^h \cos(\omega\tau) d\mu(\tau) &= \int_0^{2E} \cos(\omega\tau) d\mu(\tau) \\ &= \int_0^E \Big(\cos(\omega\tau) + \cos(\omega(2E-\tau)) \Big) d\mu(\tau). \end{aligned}$$

Now, let $0 \leq \delta < 1$. If $\omega = \delta \frac{\pi}{2E}$ then $0 \leq \omega < \frac{\pi}{2E}$. Accordingly,

$$\begin{aligned} \cos\left(\delta \frac{\pi}{2E}\tau\right) + \cos\left(\delta\frac{\pi}{2E}(2E-\tau)\right) &= 2\cos\left(\delta\frac{\pi}{2}\right)\cos\left(\delta\frac{\pi}{2E}(E-\tau)\right) \\ &> 2\cos\left(\delta\frac{\pi}{2}\right)\cos\left(\delta\frac{\pi}{2}\right) > 0. \end{aligned} \tag{28}$$

We exploited that $0 \leq \delta\frac{\pi}{2E}(E-\tau) \leq \delta\frac{\pi}{2}$, since $\tau \in [0,E]$. That is, the function $\cos\left(\delta\frac{\pi}{2E}(E-\tau)\right)$ is monotonically decreasing if $\tau \in [0,E]$.

It follows that statement 2 is valid, since we integrate a positive function with respect to a monotone increasing function. □

We remark that the case of $E = 0$ occurs only if

$$\mu(x) = \begin{cases} 1, & if\ x \geq 0; \\ 0, & if\ x < 0. \end{cases}$$

In this case

$$\left| \int_0^h \cos(\omega\tau) d\mu(\tau) \right| = |\cos(\omega E)| = 1.$$

Lemma 3.6. *Let $\mu : \mathbb{R} \to \mathbb{R}$ in (2) be symmetric about its mean $E > 0$. Then $\Omega \subseteq \hat{\Omega}$, where $\hat{\Omega}$ is the zero-set of (2).*

Proof. Let $\alpha_1, \dots, \alpha_n \in \mathbb{R}^+$, $\sum_{i=1}^n \alpha_i = \frac{1}{2}$ and

$$\mu(E - \theta_i) - \mu(E - \theta_i - 0) = \mu(E + \theta_i) - \mu(E + \theta_i - 0) := \alpha_i,$$

where $\theta_1, \dots, \theta_n \in [0, E]$ and let

$$\mu(\tau) = \sum_{i : E - \theta_i \leq \tau} \alpha_i + \sum_{i : E + \theta_i \leq \tau} \alpha_i, \quad \tau \in [0, 2E].$$

Notice that $\int_0^{2E} \tau d\mu(\tau) = E$ and μ is symmetric about its mean.

Since

$$\begin{aligned} \hat{S}(\omega) &:= \sum_{i=1}^n \left(\alpha_i \left(\sin\left(\omega\left(E - \theta_i\right)\right) + \sin\left(\omega\left(E + \theta_i\right)\right)\right)\right) \\ &= 2\sin(\omega E) \sum_{i=1}^n \alpha_i \cos(\omega\theta_i). \end{aligned} \tag{29}$$

The last expression proves the case that μ is a step function in (2).

Since, as we have seen in the proof of Lemma 3.5, an arbitrary function $\mu : \mathbb{R} \to \mathbb{R}$ can be approximated by step functions, the proof is complete. □

Given the statement of Lemma 3.6, we define the intervals $\tilde{I}_k^+ = \bigcup_{l=1}^i I_{k,l}^+$ and $\tilde{I}_k^- = \bigcup_{m=1}^j I_{k,m}^-$. We recall that $I_{k,l}^+$ and $I_{k,m}^-$, $1 \leq l \leq i$, $1 \leq m \leq j$ denotes the subintervals in I_k such that on them we have $S(\omega) > 0$ and $S(\omega) < 0$, respectively, where i and j are the (k-dependent) numbers of those subintervals. The relative positions of curves definied via the functions $a(\omega)$ and $b(\omega)$ could be quite complicated, but the following lemma shows an important feature of them. To formulate it we introduce the notation that, for an arbitrary function $\Gamma(x) : I \to (f(x), g(x)) \in \mathbb{R}^2$, the symbol $|\Gamma|$ denotes the function $|\Gamma(x)| : I \to (f(x), |g(x)|) \in \mathbb{R}^2$.

Lemma 3.7. *Let $\mu : \mathbb{R} \to \mathbb{R}$ in (2) be symmetric about its mean $E > 0$. Then $\Gamma_0 \prec |\Gamma_{k,l}^+|$ and $\Gamma_0 \prec |\Gamma_{k,m}^-|$ on $\tilde{I}_k^+$ and $\tilde{I}_k^-$, respectively, for $1 \leq l \leq i$, $1 \leq m \leq j$.*

Proof. Let μ be as in the proof of Proposition 3.6. Then, since

$$\begin{aligned} \hat{C}(\omega) &:= (\alpha_i(\cos(\omega(E - \theta_i)) + \cos(\omega(E + \theta_i)))) \\ &= 2\cos(\omega E) \sum_{i=1}^n \alpha_i \cos(\omega\theta_i) \end{aligned} \tag{30}$$

and because of (29), we obtain the functions

$$\hat{a}(\omega) = -\omega \frac{\cos(\omega E)}{\sin(\omega E)} = a(\omega)$$

$$\hat{b}(\omega) = \frac{\omega}{2\sin(\omega E)\sum_{i=1}^{n}\alpha_i\cos(\omega\theta_i)} = \frac{b(\omega)}{\sum_{i=1}^{n}\alpha_i\cos(\omega\theta_i)}.$$

That is,

$$\Gamma_{k,l}^{\pm} = \left\{\left(-\omega\frac{\cos(\omega E)}{\sin(\omega E)}, \frac{\omega}{2\sin(\omega E)\sum_{i=1}^{n}\alpha_i\cos(\omega\theta_i)}\right)\Big|\omega \in I_{k,l}^{\pm}\right\}.$$

Since $\sum_{i=1}^{n}\alpha_i\cos(\omega\theta_i) \leq 1$ thus $|\hat{b}(\omega)| > b(\omega),\ \omega \in I_{k,l}^{\pm}$. Further, if $I_{k,l}^{\pm} = (\omega_L, \omega_R)$ then

$$\lim_{\omega\to\omega_L}|\hat{b}(\omega)| = \lim_{\omega\to\omega_R}|\hat{b}(\omega)| = \infty$$

when $\omega_L \neq 0$.

Thus $\Gamma_k^+ \prec \Gamma_{k,l}^+$, $\Gamma_k^+ \prec |\Gamma_{k,m}^-|$, $\Gamma_{k,m}^- \prec \Gamma_k^-$ and $|\Gamma_k^-| \prec \Gamma_{k,m}^-$, on the associated intervals. Notice that k, l, m were arbitrary. Thus, because of the fact that $\Gamma_0 \prec \Gamma_k^+$ and $\Gamma_k^- \prec \Gamma_0,\ k \in \mathbb{N}$, our statement is valid whenever μ is a step function.

Note that

$$\lim_{\omega\to 0+} a(\omega) = \lim_{\omega\to 0+}\hat{a}(\omega) = -\lim_{\omega\to 0+}\hat{b}(\omega) = \lim_{\omega\to 0+} b(\omega) = \frac{1}{E}.$$

Again as we have seen in the proof of Lemma 3.5, an arbitrary function $\mu : \mathbb{R} \to \mathbb{R}$ can be approximated by step functions, which completes the proof. □

We are now able to state and prove our main result.

Theorem 3.8. *Let us fix a number $E > 0$ and let let $b > |a|$. Suppose that the the zero solution $x \equiv 0$ of the mean-delay equation (3) is asymptotically stable for a given pair (a, b) of parameters. Then the zero solution $x \equiv 0$ of the distributed-delay equation (2) is asymptotically stable for any distribution function μ that is symmetric about the fixed expectation value E.*

Proof. By Proposition 3.4 the number of the characteristic roots with positive real part is zero on the line $b = a$ for $a > 0$. This number can change by crossing one of the curves $\Gamma_{k,l}^+,\ k = 1, 2, \ldots,\ 1 \leq l \leq i$ or $\Gamma_{k,m}^-, k = 1, 2, \ldots,\ 1 \leq m \leq j$. Because of Lemma 3.7, this crossing is above Γ_0, thus the stability region of (3) is contained in the stability region of (2). In other words, the stability of the zero solution of (3) implies that of (2). □

As an immediate consequence of Theorem 3.8 in combination with Proposition 3.1 we obtain the following result.

Theorem 3.9. [2] *Let μ be a monotone increasing, right-continuous function with properties (A1) and (A2). Furthermore, let μ be symmetric about its mean E. Then, the trivial solution $x \equiv 0$ of (2) is asymptotically stable if*

$$E < \frac{\arccos(\frac{-a}{b})}{\sqrt{b^2 - a^2}},\ \textit{where } b > |a|.$$

This result is Statement 2 of Theorem 4.0.5 of [2]. In the proof in [2] the statement of Theorem 3.8 was implicitely used but, as far as we know, it had not been proved anywhere.

4. The case of second-order equations

We now consider stability properties of the second-order equations (5) and (6), where we use the same notation for the different functions $a(\omega)$ and $b(\omega)$, $C(\omega)$, $S(\omega)$ and $\Gamma_k^{\pm}$ as before. The characteristic function of (5) is

$$h(s): \ \mathbb{C} \to \mathbb{C}, \ s \mapsto s^2 + s + a + b\int_0^h e^{-s\tau} d\mu(\tau) \tag{31}$$

and its characteristic equation is

$$s^2 + s + a + b\int_0^h e^{-s\tau} d\mu(\tau) = 0, \ s \in \mathbb{C}. \tag{32}$$

Furthermore, the zero-set corresponding to the function $S(\omega)$ in (5) can be defined as before, thus

$$a(\omega): I_k \to \mathbb{R}, \ \omega \mapsto \omega^2 - \omega\frac{C(\omega)}{S(\omega)}, \tag{33}$$

$$b(\omega): I_k \to \mathbb{R}, \ \omega \mapsto \frac{\omega}{S(\omega)}. \tag{34}$$

We can now draw the stability region of the mean-delay equation (6). Figure 2 shows an example where the relevant parts of curves $\Gamma_k^{\pm}$ for $k = 0, 1, 2$ and the line $b = -a$ of (6) are plotted for $E = 1$. The number of unstable characteristic roots is again indicated for each region. The figure shows that the boundary of the grey stability region consists of segments of the curves $\Gamma_k^{\pm}, \ k \in \mathbb{N}$. This stability region of a second order equation is well known and has been described, for example, in [3, 24].

The question is now what can be said about the stability region of the distributed-delay equation (5) with mean E from the knowledge of the stability region of the mean-delay equation (6). Indeed, it would be very useful to have a statement analogous to Theorem 3.8, also in the second-order case. For instance in a hybrid test with more than one actuator, it would be helpful to start a test by making use of stability information obtained from the system subject to the fixed mean delay, which could be measured experimentally.

As the following example shows, it is not true in general that the stability region of the mean-delay equation (6) is contained in the stability region of the distributed-delay equation (5) — even when the delay distribution function μ is restricted to the case of a symmetric distributions with two delays. Consider the distributed-delay equation

$$\ddot{x}(t) = -\dot{x}(t) - ax(t) - b\left(\frac{1}{2}x(t-\tau_1) + \frac{1}{2}x(t-\tau_2)\right), \tag{35}$$

where $\tau_1 = 0.55$ and $\tau_1 = 1.45$, so that we have a mean of $E = 1$ in (6). For the specific choice of $a = 2\pi^2$ and $b = 0.58 \cdot 2\pi^2$ one computes that characteristic roots with the largest real part of (6) are $\lambda_{1,2} \approx -0.017896 \pm 2.899378i$, meaning that its zero solution is stable. On the other hand, the right-most eigenvalues of (35) are $\lambda_{1,2} \approx 0.040907 \pm 4.528637i$, that is, the zero solution of the distributed-delay equation (35) is unstable. Hence, this specific choice of the parameters a and b provides a counter example to a general stability statement for second-order equations (similar to to the one formulated in Theorem 3.8 for the first-order case).

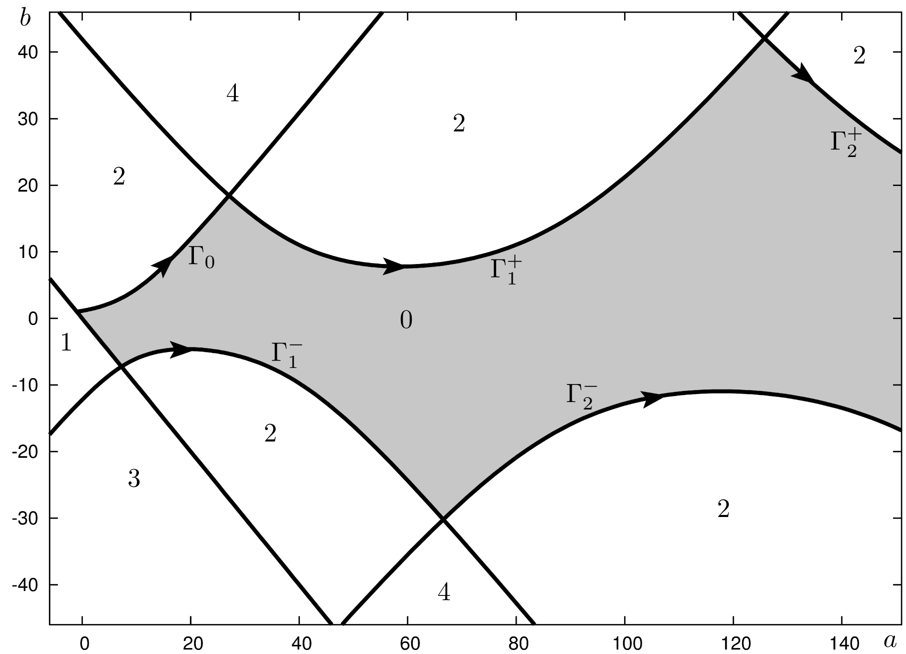

FIGURE 2. The curves $\Gamma_k^{\pm}$ for $k = 0, 1, 2$ and the line $b = -a$ of (6) for $E = 1$. The direction of increasing ω is indicated by the arrows, and the number of characteristic roots with positive real parts are indicated in each region; the shaded part of the image is the stability region of the zero solution.

To obtain a more global view of how the stability of the two equations (35) and (6) with $E = 1$ are related, Figure 3 shows the boundaries of their stability regions in the (a, b)-plane as a solid and a dashed curve, respectively. As can be seen, the two stability boundaries cross each other, so that stability for (6) with $E = 1$ does indeed not imply stability for (35) throughout the (a, b)-plane.

Figure 3 also shows that it would be difficult to make a general statement of how the two stability boundaries relate to each other as a function of, say, the variance of the distribution. Furthermore, our example features only two delays, and it appears that it would be very hard, if not impossible, to say anything in general about the relative positions of the stability boundaries of (5) and (6) with $E = 1$ when the delay is subject to a more general distribution function, even if it is a symmetric one.

Despite these difficulties, Figure 3 suggest that there is always a first point (in terms of the value of a) where the two boundary curves intersect, such that the stability region initially grows when the delay is distributed symmetrically. This observation can be formalized as follows.

Proposition 4.1. *Let $\mu : \mathbb{R} \to \mathbb{R}$ in (5) be symmetric about its mean E. Furthermore, let $S(\omega)$, $\hat{S}(\omega)$, $I_k^+, \tilde{I}_k^+, I_k^-, \tilde{I}_k^-,\ k \in \mathbb{N}$ be the functions and domains associated*

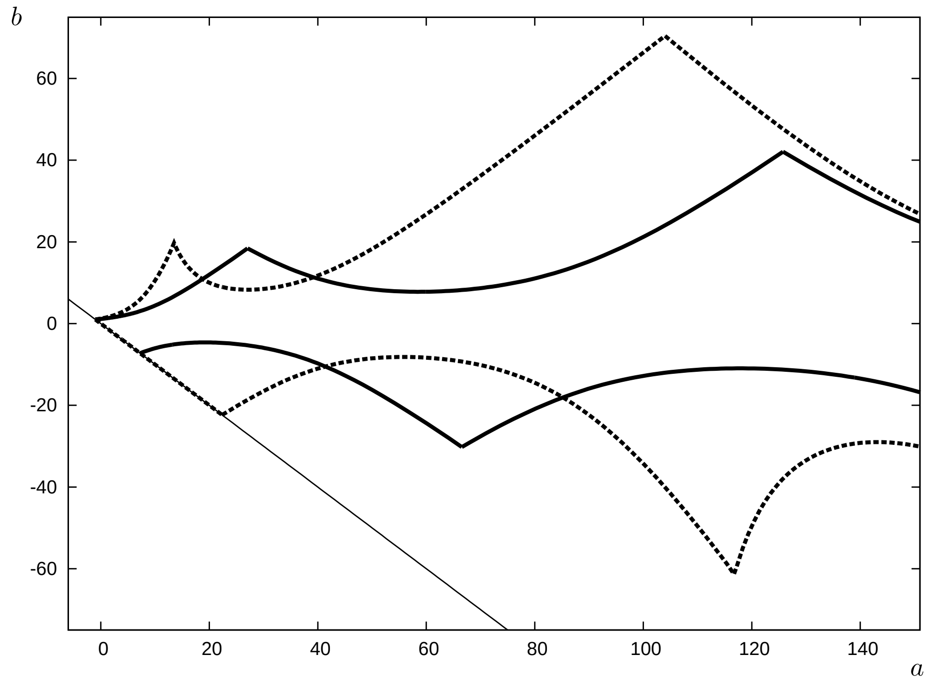

FIGURE 3. The solid curve is the boundary of the stability region of (6) for $E = 1$ (compare with Figure 2), and the dashed curve is the boundary of the stability region of (35).

with (6) *and* (5)*, respectively, and let*

$$k^* = \min_{k\in\mathbb{N}}\{k : S(\omega)\hat{S}(\omega) < 0, \omega \in \tilde{I}_k^+\}.$$

Then

$$\Gamma_k^+ \prec \Gamma_{k,l}^+$$

and

$$\Gamma_{k,m}^- \prec \Gamma_k^-,$$

for $k = 0 \ldots k^* - 1,\ 1 \le l \le i,\ 1 \le m \le j.$

Proof. Again, it is enough to show the existence of k^* for step functions. With the notation as in the proof of Lemma 3.6, the existence of k^* follows from the fact that

$$S\left(\omega + \frac{2k\pi}{E}\right)\hat{S}\left(\omega + \frac{2k\pi}{E}\right) = 2\sin^2(\omega E)\sum_{i=1}^{n}\alpha_i \cos\left(\left(\omega + \frac{2k\pi}{E}\right)\theta_i\right) \tag{36}$$

The remaining part of the proof of the statement is not given here because it is a slight and natural modification of the proof of the corresponding statement Lemma 3.7 for the first-order case. Namely, after applying the trigonometric identities used in (29) and (30) one follows the exact same steps. □

From an application point of view, it would be important to know how the value of k^* depends on the distribution function in question. However and unfortunately, the mean E of the delay distribution does not contain enough information to describe this dependence. That is, after fixing the value of E one may find different values of k^* for different distributions. The problem of finding properties of (symmetric) distribution functions μ, such as higher moments, that determine the corresponding value of k^* remains a challenge for further research; see also [6].

Another question is whether the stability boundaries of equations (6) and (5) will actually always intersect for any symmetric μ. Based on experience with a number of examples, we conjecture that this is indeed the case, which can be formulated as follows.

Conjecture 4.2. *Let k^* be as as in Proposition 4.1. Then there are $1 \leq l^* \leq i$ and $1 \leq m^* \leq j$ such that some part of Γ^+_{k*,l^*} is above and some part is below Γ^+_{k*}. Furthermore, some part of Γ^-_{k*,m^*} is above and some part is below Γ^-_{k*}.*

Conjecture 4.2 implies that there exists a first (with respect to a) intersection point (a^*, b^*) of the two stability boundaries, that is, there is no intersection for $a < a^*$. Therefore, if we can prove Conjecture 4.2, and with Proposition 4.1, we would obtain the following statement for the second-order case.

Conjecture 4.3. *For a given fixed number $E > 0$, let a^* be the first intersection point of the stability boundaries of (6) and (5). Suppose that the trivial solution $x \equiv 0$ of equation (6) is asymptotically stable for a given pair (a, b) of parameters with $a < a^*$. Then the trivial solution $x \equiv 0$ of equation (5) at (a, b) is asymptotically stable for any distribution function μ that is symmetric about the fixed mean E.*

5. Conclusion

We considered whether symmetric delay distribution preserves the stability of the correpsonding equation where the delay is fixed and given by the mean of the distribution. We showed that for first-order equations such a general result is indeed true, which completes the proof of a stability criterion in [2]. For second-order equations, on the other hand, an example showed that the stability region of the (symmetrically) distributed-delay equation is generally not a subset of the stability region of the mean-delay equation. We conjecture that the boundaries of the respective stability regions always intersect. This would imply a maximal generalization of preserved stability under symmetric delay distribution up to the point of first intersection. A more general investigation of the effect of the delay distribution on second-order equations remains an interesting challenge for future research. To be more specific, it would be desirable to identify (in terms of further generic properties of the distribution function) what causes crossings of the two stability boundaries. The goal would be to formulate conditions on the distribution that allow one to exclude such intersection from a desired part of parameter space.

More generally, our results provides an easily verifiable condition for stability of a delay differential equation with symmetric distribution function. We expect this to be of interest in applications, for example, to ensure that a hybrid-test experiment is started and run in its stable region (away from, for example, oscillatory instabilities). In the second order case, some extra investigations are required to locate the part of the stability chart where stability actually increases. We further envisage that the results obtained here could be used in conjunction with numerical continuation techniques [22] to evaluate the stability of distributed-delay models arising in concrete applications.

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